LETTER TO THE EDITOR

Finite geometries and diffractive orbits in isospectral billiards

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Abstract
Several examples of pairs of isospectral planar domains have been produced
in the two-dimensional Euclidean space by various methods. We show that
all these examples rely on the symmetry between points and blocks in finite
projective spaces; from the properties of these spaces, one can derive a relation
between Green's functions as well as a relation between diffractive orbits in
isospectral billiards.

PACS numbers: 02.70.Hm, 03.65.Nk, 03.65.Ge, 05.45.−a

It is almost 40 years now that Mark Kac addressed his famous question "Can one hear the shape
of a drum" [1]. The problem was to know whether there exist isospectral domains, that is, non-
isometric bounded regions of space for which the sets $\{E_n, n \in \mathbb{N}\}$ of solutions of the stationary
Schrödinger equation $(\Delta + E)\Psi = 0$, with some specified boundary conditions, would
coincide. Instances of isospectral pairs have finally been found for Riemannian manifolds
[2, 3], and more recently for two-dimensional Euclidean connected compact domains (we
will call such planar domains 'billiards') [4]. The proof of isospectrality was based on
Sunada's theorem, which allows one to construct isospectral pairs from groups related by
the so-called 'almost conjugate' property [5]. The two billiards given in [4] to illustrate
the existence of isospectral billiards are represented in figure 1(*a*): they are made of seven
copies of a base tile (here a right isosceles triangle) unfolded with respect to its edges in
two different ways. It turns out that in this example, isospectrality can be proved directly by
giving an explicit linear map between the two domains [6]. Eigenfunctions in one billiard
can be expressed as a linear superposition of eigenfunctions of the other billiard, in such a
way that boundary conditions are fulfilled; isospectrality is ensured by the linearity of the
Schrödinger equation. This 'transplantation' method allowed one to generalize the example of
[4]: any triangle can replace the base triangle of figure 1(*a*), what matters is the way the
initial base tile is unfolded (see figure 1(*b*)). Further examples of isospectral billiards were
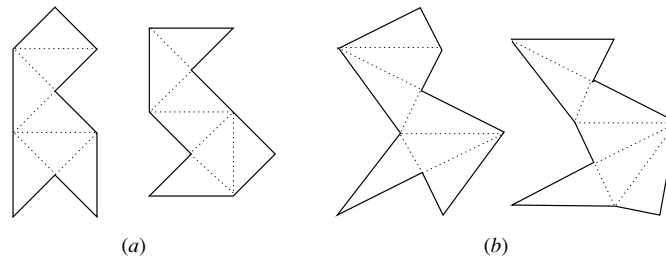
Figure 1. (*a*) Two isospectral billiards with a triangular base shape. (*b*) A pair of isospectral billiards constructed from the same unfolding rules as (*a*).

obtained by applying Sunada's theorem to reflection groups in the hyperbolic upper-half plane [9]; examples of billiards made of 7, 13, 15 or 21 triangular tiles were given. Later on, it was shown that the transplantation method produces exactly the same examples and provides a proof for the iso-length-spectrality of these billiards (i.e., the set of lengths of periodic orbits is identical) [10]. A similar transplantation technique, together with results on Green's functions of polygonal billiards, was applied to show that there is no such equality of the lengths for diffractive orbits (that is, the lines joining two vertices); a more involved relation was found in [12].

The purpose of this letter is to show that all known examples of isospectral pairs, given by [9] or [10], can be obtained by considering finite projective spaces (FPSs): the transplantation map between two isospectral billiards can be taken as the incidence matrix of an FPS. This construction from FPSs indicates that the deep origin of isospectrality in Euclidean spaces is point–block duality. It also leads to a relation between Green's functions, and to a correspondence between diffractive orbits in isospectral billiards.

Isospectral billiards. All known isospectral billiards can be obtained by unfolding triangle-shaped tiles [9, 10]. The way the tiles are unfolded can be specified by three permutation matrices $M^{(\mu)}$, $1 \leqslant \mu \leqslant 3$, associated with the three sides of the triangle: $M^{(\mu)}_{ij} = 1$ if tiles i and j are glued by their side μ (and $M^{(\mu)}_{ii} = 1$ if the side μ of tile i is the boundary of the billiard), and 0 otherwise [10–12]. Following [10], one can sum up the action of the $M^{(\mu)}$ in a graph with coloured edges: each copy of the base tile is associated with a vertex, and vertices i and j, $i \neq j$, are linked by an edge of colour μ if and only if $M^{(\mu)}_{ij} = 1$ (see figure 2). In the same way, in the second member of the pair, the tiles are unfolded according to permutation matrices $N^{(\mu)}$, $1 \leqslant \mu \leqslant 3$. Two billiards are said to be transplantable if there exists an invertible matrix T (the *transplantation matrix*) such that $\forall \mu \; T M^{(\mu)} = N^{(\mu)} T$. One can show that transplantability implies isospectrality (if the matrix T is not merely a permutation matrix, in which case the two domains would just have the same shape). The underlying idea is that if $\psi^{(1)}$ is an eigenfunction of the first billiard and $\psi^{(1)}_i$ its restriction to triangle i, then one can build an eigenfunction $\psi^{(2)}$ of the second billiard by taking $\psi^{(2)}_i = \sum_j T_{ij} \psi^{(1)}_j$. Obviously $\psi^{(2)}$ verifies the Schrödinger equation; it can be checked from the commutation relations that the function is smooth at all edges of the triangles, and that boundary conditions at the boundary of the billiard are fulfilled [10].

Suppose that we want to construct a pair of isospectral billiards, starting from any polygonal base shape. Our idea is to start from the transplantation matrix, and choose it in such a way that the existence of commutation relations $T M^{(\mu)} = N^{(\mu)} T$ for some permutation matrices $M^{(\mu)}$, $N^{(\mu)}$ will be known *a priori*. As we will see, this is the case if T is taken as the
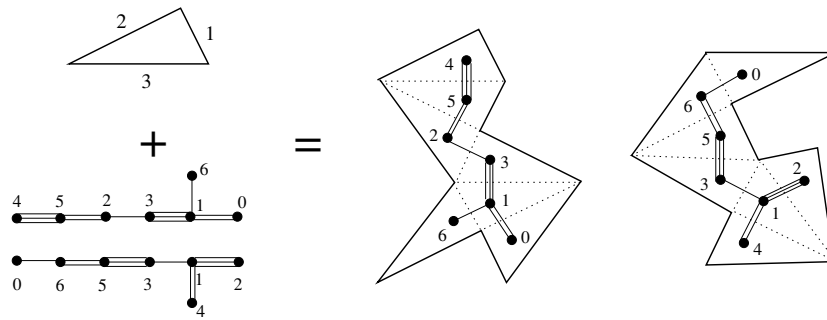
Figure 2. The graphs corresponding to a pair of isospectral billiards: if we label the sides of the triangle by $\mu = 1, 2, 3$, the unfolding rule by symmetry with respect to side μ can be represented by edges made of μ braids in the graph. From a given pair of graphs, one can construct infinitely many pairs of isospectral billiards by applying the unfolding rules to any triangle. Note that a different labelling of the tiles would just induce a permutation of the labellings of points and blocks in the Fano plane.
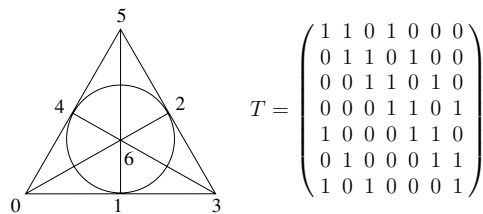

$$T = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Figure 3. The Fano plane $PG(2, 2)$ and its corresponding incidence matrix T. The Fano plane has $(q^3 - 1)/(q - 1) = 7$ points and 7 lines; each line contains $q + 1 = 3$ points and each point belongs to three lines. Any pair of points belongs to one and only one line.

incidence matrix of an FPS; the matrices $M^{(\mu)}$ and $N^{(\mu)}$ are then permutations on the points and the hyperplanes of the FPS.

Finite projective spaces. For $n \geqslant 2$ and $q = p^h$ a power of a prime number, consider the $(n + 1)$-dimensional vector space $\mathbb{F}_q^{n+1}$, where $\mathbb{F}_q$ is the finite field of order q. The *finite projective space* $PG(n, q)$ of *dimension n* and *order q* is the set of subspaces of $\mathbb{F}_q^{n+1}$: the points of $PG(n, q)$ are the one-dimensional subspaces of $\mathbb{F}_q^{n+1}$, the lines of $PG(n, q)$ are the two-dimensional subspaces of $\mathbb{F}_q^{n+1}$, and more generally $(d+1)$-dimensional subspaces of $\mathbb{F}_q^{n+1}$ are called *d-spaces* of $PG(n, q)$; the $(n - 1)$-spaces of $PG(n, q)$ are called hyperplanes or *blocks*. A *d*-space of $PG(n, q)$ contains $(q^{d+1} - 1)/(q - 1)$ points. In particular, $PG(n, q)$ has $(q^{n+1} - 1)/(q - 1)$ points. It also has $(q^{n+1} - 1)/(q - 1)$ blocks [8]. As an example, figure 3 shows the finite projective plane (FPP) of order $q = 2$, or Fano plane, $PG(2, 2)$.

An (N, k, λ)-*symmetric balanced incomplete block design* (SBIBD) is a set of N points, belonging to N subsets (or *blocks*); each block contains k points, in such a way that any two points belong to exactly λ blocks, and each point is contained in k different blocks [13]. One can show that $PG(n, q)$ is an (N, k, λ)-SBIBD with $N = (q^{n+1}-1)/(q-1)$, $k = (q^n-1)/(q-1)$ and $\lambda = (q^{n-1} - 1)/(q - 1))$. For example, the Fano plane is a $(7, 3, 1)$-SBIBD. The points and the blocks can be labelled from 0 to $N - 1$. For any (N, k, λ)-SBIBD one can define an $N \times N$ *incidence matrix* T describing to which block each point belongs. The entries T_{ij} of the matrix are $T_{ij} = 1$ if point j belongs to line i, 0 otherwise. The matrix T verifies the relation

$TT^t = \lambda J + (N-k)\lambda/(k-1)I$, where J is the $N \times N$ matrix with all entries equal to 1 and I is the $N \times N$ identity matrix [13]. In particular, the incidence matrix of $PG(n,q)$ verifies

$$TT^t = \lambda J + (k-\lambda)I \qquad (1)$$

with k and λ as given above. For example, the incidence matrix of the Fano plane given in figure 3 corresponds to a labelling of the lines such that line 0 contains points 0, 1, 3, and line 1 contains points 1, 2, 4, etc.

A *collineation* of an FPS is a bijection that preserves incidence, that is, a permutation of the points that takes d-spaces to d-spaces (in particular, it takes blocks to blocks). Any permutation σ on the points can be written as an $N \times N$ *permutation matrix* M defined by $M_{i\sigma(i)} = 1$ and the other entries equal to zero. If M is a permutation matrix associated with a collineation, then there exists a permutation matrix N such that

$$TM = NT. \qquad (2)$$

In words, (2) means that permuting the columns of T (i.e., the blocks of the space) under M is equivalent to permuting the rows of T (i.e., the points of the space) under N. The commutation relation (2) is related to an important feature of projective geometry, the so-called 'principle of duality' [8]. This principle states that for any theorem which is true in an FPS, the dual theorem obtained by exchanging the words 'point' and 'block' is also true. As we will see now, this symmetry between points and blocks in FPSs is the central reason which accounts for the known pairs of isospectral billiards.

Let us consider an FPS $\mathcal{P}$ with incidence matrix T. With each block in $\mathcal{P}$ we associate a tile in the first billiard, and with each point in $\mathcal{P}$ we associate a tile in the second billiard. If we choose permutations $M^{(\mu)}$ in the collineation group of $\mathcal{P}$, then the commutation relation (2) will ensure that there exist permutations $N^{(\mu)}$ verifying $TM^{(\mu)} = N^{(\mu)}T$. These commutation relations imply transplantability, and thus isospectrality, of the billiards constructed from the graphs corresponding to $M^{(\mu)}$ and $N^{(\mu)}$.

If the base tile has r sides, we need to choose r elements $M^{(\mu)}, 1 \leqslant \mu \leqslant r$, in the collineation group of $\mathcal{P}$. This choice is constrained by several factors. Since $M^{(\mu)}$ represents the reflection of a tile with respect to one of its sides, it has to be of order 2 (i.e. an involution). In order that the billiards be connected, no point should be left out by the matrices $M^{(\mu)}$ (in other words, the graph associated with the matrices $M^{(\mu)}$ should be connected). Finally, if we want the base tile to be of any shape, there should be no loop in the graph. We now need to characterize collineations of order 2.

Collineations of finite projective spaces. Let $q = p^h$ be a power of a prime number. Each point P of $PG(n,q)$ is a one-dimensional subspace of $\mathbb{F}_q^{n+1}$, spanned by some vector v. We write $P = P(v)$.

An *automorphism* is a bijection of the points $P(v_i)$ of $PG(n,q)$ obtained by the action of an automorphism of $\mathbb{F}_q$ on the coordinates of the v_i. If $q = p^h$, the automorphisms of $\mathbb{F}_q$ are $t \mapsto t^{p^i}, 0 \leqslant i \leqslant h-1$.

A *projectivity* is a bijection of the points $P(v_i)$ of $PG(n,q)$ obtained by the action of a linear map L on the v_i. There are $q-1$ matrices tL, with $t \in \mathbb{F}_q \setminus \{0\}$ and $L \in GL_{n+1}(\mathbb{F}_q)$ (the group of $(n+1) \times (n+1)$ invertible matrices with coefficients in $\mathbb{F}_q$), yielding the same projectivity $P(v_i) \mapsto P(Lv_i)$.

The *fundamental theorem of projective geometry* states that any collineation of $PG(n,q)$ can be written as the composition of a projectivity by an automorphism [8]. The converse is obviously true since projectivities and automorphisms are collineations. Therefore the set of all collineations is obtained by taking the composition of all the non-singular $(n+1) \times (n+1)$ matrices with coefficients in $\mathbb{F}_q$ by all the h automorphisms of $\mathbb{F}_q$.

The collineation group of $PG(n, q)$ has $\left[h \prod_{k=0}^{n}(q^{n+1} - q^k)\right]/(q - 1)$ elements, among which we only want to consider elements of order 2. In the case of FPPs ($n = 2$), there are various known properties characterizing collineations of order 2. A *central collineation*, or *perspectivity*, is a collineation fixing each line through a point C (called the centre). By 'fixing' we mean that the line is invariant but the points can be permuted within the line. One can show that the fixed points of a non-identical perspectivity are the centre itself and all points on a line (called the axis), while the fixed lines are the axis and all lines through the centre. If the centre lies off the axis a perspectivity is called a *homology* (and has $q + 2$ fixed points), whereas if the centre lies on the axis it is called an *elation* (and has $q + 1$ fixed points). Perspectivities in dimension $n = 2$ have the following properties [14]:

Proposition 1. *A perspectivity of order 2 of an FPP of order q is an elation or a homology according to whether q is even or odd.*

Proposition 2. *A collineation of order 2 of an FPP of order q is a perspectivity if q is not a square; it is a collineation fixing all points and lines in a subplane if q is a square (a subplane is a subset of points having all the properties of an FPP).*

When the order of the FPP is a square, there is the following result [8]:

Proposition 3. $PG(2, q^2)$ *can be partitioned into* $q^2 - q + 1$ *subplanes* $PG(2, q)$.

Generating isospectral pairs. Let us assume that we are looking for a pair of isospectral billiards with $N = (q^3 - 1)/(q - 1)$ copies of a base tile having the shape of an r-gon, $r \geqslant 3$. We need to find r collineations of order 2 such that the associated graph is connected and without any loop. Such a graph connects N vertices and thus requires $N - 1$ edges. From propositions 1–3, we can deduce the number s of fixed points of a collineation of order 2 for any FPP. Since a collineation is a permutation, it has a cycle decomposition as a product of transpositions. For permutations of order 2 with s fixed points, there are $(N - s)/2$ independent transpositions in this decomposition. Each transposition is represented by an edge in the graph. As a consequence, q, r and s have to fulfil the following condition: $r(q^2 + q + 1 - s)/2 = q^2 + q$. Let us examine the various cases.

If q is even and not a square, propositions 1 and 2 imply that any collineation of order 2 is an elation and therefore has $q + 1$ fixed points. Therefore, q and r are constrained by the relation $rq^2/2 = q^2 + q$. The only integer solution with $r \geqslant 3$ and $q \geqslant 2$ is ($r = 3, q = 2$). These isospectral billiards correspond to $PG(2, 2)$ and will be made of $N = 7$ copies of a base triangle.

If q is odd and not a square, propositions 1 and 2 imply that any collineation of order 2 is a homology and therefore has $q + 2$ fixed points. Therefore, q and r are constrained by the relation $r(q^2 - 1)/2 = q^2 + q$. The only integer solution with $r \geqslant 3$ and $q \geqslant 2$ is ($r = 3, q = 3$). These isospectral billiards correspond to $PG(2, 3)$ and will be made of $N = 13$ copies of a base triangle.

If $q = p^2$ is a square, propositions 2 and 3 imply that any collineation of order 2 fixes all points in a subplane $PG(2, p)$ and therefore has $p^2 + p + 1$ fixed points. Therefore, p and r are constrained by the relation $r(p^4 - p)/2 = p^4 + p^2$. There is no integer solution with $r \geqslant 3$ and $q \geqslant 2$. However, one can look for isospectral billiards with loops: this will require the base tile to have a shape such that the loop does not make the copies of the tiles come on top of each other when unfolded. If we tolerate one loop in the graph describing the isospectral billiards, then there are N edges in the graph instead of $N - 1$ and the equation for p and r becomes $r(p^4 - p)/2 = p^4 + p^2 + 1$, which has the only integer solution ($r = 3, p = 2$).

These isospectral billiards correspond to $PG(2, 4)$ and will be made of $N = 21$ copies of a base triangle.

We can now generate all possible pairs of isospectral billiards whose transplantation matrix is the incidence matrix of $PG(2, q)$, with r and q restricted by the previous analysis. All pairs must have a triangular base shape ($r = 3$). $PG(2, 2)$ provides 3 pairs (made of 7 tiles), $PG(2, 3)$ provides 9 pairs (made of 13 tiles), $PG(2, 4)$ provides 1 pair (made of 21 tiles). It turns out that the pairs obtained here are exactly those obtained by [9] and [10] by other methods. Let us now consider spaces $PG(n, q)$ of higher dimensions. The smallest one is $PG(3, 2)$, which contains 15 points. Since the base field for $PG(3, 2)$ is $\mathbb{F}_2$, the fundamental theorem of projective geometry states that the collineation group of $PG(3, 2)$ is essentially the group $GL_4(\mathbb{F}_2)$ of 4×4 non-singular matrices with coefficients in $\mathbb{F}_2$. Generating all possible graphs from the 315 elements of order 2 in $GL_4(\mathbb{F}_2)$, we obtain four pairs of isospectral billiards with 15 triangular tiles, which completes the list of all pairs found in [9] and [10].

Our method explicitly gives the transplantation matrix T for all these pairs: each one is the incidence matrix of an FPS. The transplantation matrix explicitly provides the mapping between eigenfunctions of both billiards. The inverse mapping is given by $T^{-1} = (1/q^{n-1})(T^t - (\lambda/k)J)$. For all pairs, isospectrality can therefore be explained by the symmetry between points and blocks in FPSs. We do not know if it is possible to find isospectral billiards for which isospectrality would not rely on this symmetry.

Our construction furthermore allows us to generalize the results we obtained in [12], where a relation between the Green's functions of the billiards in an isospectral pair was derived. A similar relation can be found for all other pairs obtained by point–block duality. Let $M^{(\mu)}$ and $N^{(\mu)}$ be the matrices describing the gluing of the tiles. With any path p going from a point to another on the first billiard, one can associate the sequence $(\mu_1, \ldots, \mu_n), 1 \leqslant \mu_i \leqslant 3$, of edges of the triangle hit by the path. The matrix $M = \prod M^{(\mu_i)}$ is such that $M_{ij} = 1$ if path p can be drawn from i to j. If $N = T^{-1}MT$, relations (1) and (2) imply $\sum_{k,l} T_{ik}T_{jl}M_{kl} = \lambda + (k - \lambda)N_{ij}$. Since Green's functions can be written as a sum over all paths, the relation between the Green's functions $G^{(2)}(a, i; b, j)$ and $G^{(1)}(a, i'; b, j')$ is $\sum_{i', j'} T_{i,i'}T_{j,j'}G^{(1)}(a, i'; b, j') = (k - \lambda)G^{(2)}(a, i; b, j) + \lambda G^t(a; b)$, where G^t is the Green's function of the triangle, and a point (a, i) is specified by a tile number i and its position a inside the tile (see [12] for further detail). More precisely, the term $\sum_{k,l} T_{ik}T_{jl}M_{kl}$ can be interpreted as the number of pairs of tiles (i, j) in the first billiard such that a path identical to p can go from i to j. A given diffractive orbit going from i to j in the second billiard corresponds to a matrix N such that $N_{ij} = 1$; it is therefore constructed from a superposition of $k = (q^n - 1)/(q - 1)$ identical diffractive orbits in the first billiard. (Note that these results correspond to the Neumann boundary conditions. It is easy to obtain similar relations for the Dirichlet boundary conditions by conjugating all matrices with a diagonal matrix D with entries $D_{ii} = \pm 1$ according to whether tile i is like the initial tile or like its mirror inverse.)

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